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Instanton Effects in Supersymmetric Yang-Mills Theories on ALE Gravitational Backgrounds

MASSIMO BIANCHI, FRANCESCO FUCITO, GIANCARLO ROSSI

*Dipartimento di Fisica, Università di Roma II “Tor Vergata” and
I.N.F.N. Sezione di Roma II “Tor Vergata”, 00133 Roma, Italy*

and

MAURIZIO MARTELLINI

*Dipartimento di Fisica, Università di Milano, 20133 Milano, Italy
and Sezione I.N.F.N. dell’Università di Pavia, 27100 Pavia, Italy*

ABSTRACT

In this letter we report on the computation of instanton-dominated correlation functions in supersymmetric Yang-Mills theories on Asymptotically Locally Euclidean spaces. Following the approach of Kronheimer and Nakajima, we explicitly construct the self-dual connection on Asymptotically Locally Euclidean spaces necessary to perform such computations. We restrict our attention to the simplest case of an $SU(2)$ connection with lowest Chern class on the Eguchi-Hanson gravitational background.

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Introduction

Understanding non-perturbative phenomena is a key issue in any field theory. These effects are supposed to play a fundamental role in the explanation of confinement as well as dynamical supersymmetry (SUSY) breaking and many interesting results that go beyond perturbation theory have been recently obtained for supersymmetric Yang-Mills theory (SYM from now on) on flat [1,2,3] and curved manifolds [4,5].

Constant values for instanton dominated correlation functions may be related to topological invariants of the moduli space of YM connections on the base manifold [6,7]. When clustering applies, they give rise to the formation of chiral condensates.

The local extension of the results obtained for globally SYM theories presents formidable difficulties, the main one being the non renormalizability of the resulting quantum (super-)gravity. A way to circumvent this problem is to have the theory embedded in a suitable string theory which will act as a regulator. One thus seems to be lead to consider only the four dimensional effective field theories which emerge as low-energy limits of consistent superstring compactifications.

Euclidean supersymmetric solutions of the heterotic (or Type I) string equations of motion can be found setting to zero the fermionic fields together with their SUSY variations. For these solutions not to get corrections in (the σ -model coupling constant) α' , one is lead to impose the “standard-embedding” of the generalized spin connection into the gauge group [8,9,10,11,12]. Leaving aside solutions with non-trivial configurations of the scalar fields [13,14,15], which may be related to monopole solutions [16], one has the option of either taking a constant dilaton background [10] or a non-trivial axionic instanton [13,8,9].

In the following we will mainly concentrate on the former choice which leads to self-dual gauge connections on manifolds with self-dual curvature. We further restrict our investigation to self-dual ALE instantons. They have been completely classified by Hitchin and Kronheimer [17,18] and, as shown by Kronheimer and

Nakajima [20], admit a generalized ADHM procedure [19] which can be exploited to completely solve the problem of constructing the most general self-dual gauge connection on an ALE manifold. String propagation on ALE manifolds has been considered in [10,21]. It should be remembered that the metrics on these manifolds represent true minima of the gravitational part of the locally supersymmetric action which is obtained as a low-energy limit of the heterotic (or Type I) string. ALE instantons can thus be taken as good starting points for interesting string-inspired calculations (*e.g.* for the study of dynamical supersymmetry breaking).

The purpose of this letter is to briefly illustrate the computation of instanton-dominated correlation functions of SYM theories on ALE manifolds. For definitiveness we will consider the simplest case: an $SU(2)$ YM instanton with the lowest possible Chern class (*i.e.* $c_2 = \frac{1}{2}$) on the Eguchi-Hanson gravitational background (from now on we will refer to this case as the “minimal instanton”). A general description of the techniques underlying the Kronheimer-Nakajima (KN) construction as well as the much more complicated computations involving $SU(2)$ instantons with higher values of the second Chern class will be presented elsewhere [22]. We would like to stress here that the minimal instanton is a solution of the heterotic string equations of motion only to lowest order in α' . The constant dilaton background must be corrected order by order in α' . On the contrary the solution obtained through the standard embedding [10], which corresponds to $c_2 = \frac{3}{2}$, is not expected to receive perturbative corrections in α' and will be discussed in [22].

The four-dimensional “framed” moduli space[★] of the minimal instanton turns out to be equivalent, as a hyperkähler manifold, to the base manifold itself [20,23]. The correlation functions which saturate the chiral selection rule depend on the number of supersymmetries. For $N = 1$ it is the vacuum expectation value (*vev*) of the one-point gaugino bilinear $\text{Tr}(\lambda\lambda(x))$ which is relevant. For $N = 2$ one has to consider the one-point composite field $\text{Tr}(\phi^2(x))$. Finally, for $N = 4$, it

★ The “framed” moduli space is defined as the space of self-dual connections modulo the group of homotopically trivial gauge transformations with local support.

is simply the *vev* of the identity operator which receives contribution from the minimal instanton.

The prototype of ALE metrics is the Eguchi-Hanson (EH) gravitational background [24]

$$ds^2 = g_{\mu\nu}^{EH} dx^\mu dx^\nu = \left(\frac{r}{u}\right)^2 dr^2 + r^2(\sigma_x^2 + \sigma_y^2) + u^2 \sigma_z^2 \quad (1)$$

where σ_x , σ_y and σ_z are the left-invariant forms on $SU(2)$. The bolt singularity at $r = a$, is removed by changing the radial variable to $u = r\sqrt{1 - \left(\frac{a}{r}\right)^4}$ and identifying antipodal points. The EH background has an S^3/\mathbb{Z}_2 boundary and admits an $SU(2)_R \otimes U(1)_L$ isometry group. It is well known that the EH metric is a solution of the euclidean Einstein equations with Euler characteristic, χ , equal to 2 and Hirzebruch signature, $\tau = b_2^+ - b_2^-$, equal to 1. Coupling this solution to gauge fields via the “standard embedding”, allows to promote it to a fullfledged solution of the heterotic string equations of motion [10].

We recall that, as there are no (normalizable) anti-self-dual two-forms on the EH background, there exists exactly one self-dual two-form given by [24]

$$F = F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{2a^2}{r^4} (r dr \wedge \sigma_z + r^2 \sigma_x \wedge \sigma_y) \quad (2)$$

which may be thought as the field strength of the “monopole potential”

$$A = A_\mu dx^\mu = \frac{a^2}{r^2} \sigma_z \quad (3)$$

Gibbons and Hawking [25] generalized the metric (1) to a class of multicenter metrics with increasing Euler characteristic and Hirzebruch signature. The EH case corresponds to a “two-center” metric, the “one-center” case being diffeomorphic to flat Euclidean space. Eliminating the apparent singularities in the n -center metrics at the location of the centers, requires an identification of points under a

discrete group \mathbb{Z}_n so that the boundary of the manifold turns out to be S^3/\mathbb{Z}_n and the resulting asymptotic metric is not globally Euclidean but only Asymptotically Locally Euclidean, whence the name.

Using twistor techniques, Hitchin [17] has shown that ALE manifolds are smooth resolutions of algebraic varieties in \mathbb{C}^3 . Simple singularities admit an A-D-E classification according to which, the class of multicenter of Gibbons-Hawking may be identified with the resolution of singularities of A-type. A general construction of all ALE manifolds was worked out by Kronheimer [18]. In the Kronheimer construction, these manifolds emerge as minimal resolutions of \mathbb{C}^3/Γ , where the discrete subgroups of $SU(2)$, Γ , entering into the quotient are in one-to-one correspondence with the extended Dynkin diagrams of simply-laced simple Lie algebras.

Instanton Construction and Hyperkähler Quotient

ALE manifolds are non-compact hyperkähler manifolds, *i.e.* manifolds which admit three closed Kähler forms $\omega_{\mu\nu}^i = g_{\mu\lambda}(J^i)^\lambda{}_\nu$, where $i = 1, 2, 3$, $g_{\mu\nu}$ is the metric on the manifold and J^i are three covariantly constant complex structures satisfying the quaternionic algebra. The hyperkähler metric on an ALE space may be explicitly constructed following a standard procedure known as hyperkähler quotient [26]. This is a general method to build a hyperkähler manifold, X , starting from another one, M , admitting “triholomorphic” isometries. These isometries are generated by vector fields, v , satisfying $\mathcal{L}_v\omega^i \equiv i_v d\omega^i + d(i_v\omega^i) = 0$, where i_v denotes contraction with the components of v . Any vector field of this kind admits in fact three Killing potentials, μ_v^i , which can be thought as the hyperkähler generalization of the well-known Hamiltonian potentials corresponding to conserved quantities in classical mechanics. They can be obtained integrating the equations $d\mu_v^i = i_v\omega^i$, whenever the ω^i are closed forms.

Let M be a hyperkähler manifold of real dimension $d_M = 4m$ and H a compact group of $d_H = h$ freely acting triholomorphic isometries generated by v_a , $a = 1, \dots, h$. One may construct a submanifold, P_ζ , of dimension $d_P = d_M - 3d_H =$

$4m - 3h$, by defining

$$P_\zeta = \{p \in M : \mu_a^i(p) = \zeta_a^i, \quad i = 1, 2, 3, \quad a = 1, \dots, h\} \quad (4)$$

When ζ_a^i belongs to $\mathbb{R}^3 \times \mathcal{Z}^*$, with \mathcal{Z}^* the dual of the center of \mathcal{H} (the Lie algebra of H), the hypersurface P_ζ is preserved by the action of H . In fact P_ζ turns out to be a H -principal bundle over the hyperkähler manifold $X_\zeta = P_\zeta/H$ of dimension $d_X = d_P - d_H = d_M - 4d_H = 4(m - h)$. The coset space X_ζ is precisely the hyperkähler quotient.

More explicitly, in the Kronheimer construction of ALE instantons, one starts with the flat hyperkähler space $M = \{Q \otimes \text{End}(R)\}_\Gamma$, *i.e.* the space of Γ -invariant “doublets” of self-adjoint endomorphisms of the vector space of the regular representation R of Γ . Denoting by R_i the irreducible representations of Γ of dimension n_i , we have $R = \oplus_i R_i \otimes \bar{R}_i$ and $d_R = \sum_i n_i^2 = |\Gamma|$, where the index i is taken to run from 0 to $r - 1$, with r equal to the number of conjugacy classes in Γ . The two-dimensional representation, Q , of Γ defines the natural embedding of Γ into $SU(2)$ and it is isomorphic to the space of left-handed spinors, $Q \sim S^-$. Its conjugate is denoted by S^+ . In taking the hyperkähler quotient one identifies H with the group $\prod_i U(n_i)/U(1)$ of freely acting triholomorphic isometries commuting with the left action of Γ on R .

To the H -principal bundle, P_ζ , one may associate the so-called “tautological bundle” \mathcal{T} , a vector bundle whose typical fiber is the vector space of the regular representation. As a result of the above construction, the curvature of the natural connection on the principal bundle P_ζ over X_ζ is self-dual [26,27].

The ADHM construction [19,28] that gives rise to all self-dual connections on S^4 can also be cast in the language of the moment map and carried over to \mathbb{R}^4 . Starting from this observation Kronheimer and Nakajima brought the ADHM and the hyperkähler quotient construction of ALE spaces together [20].

The initial step in the KN construction is to give a set of “ADHM data” $\{A, B, s, t, \xi\}$ where $\xi \in M$, A and B are Γ -equivariant endomorphisms of a k -

dimensional complex vector space, V , and s, t is a pair of homomorphisms between V and an n -dimensional complex vector space W . Both V and W are Γ -modules, *i.e.* they admit the decompositions

$$V = \oplus_i V_i \otimes R_i \quad W = \oplus_i W_i \otimes R_i \quad (5)$$

where $V_i \sim \mathbb{C}^{v_i}$, $W_i \sim \mathbb{C}^{w_i}$. Therefore $k = \dim V = \sum_i n_i v_i$ and $n = \dim W = \sum_i n_i w_i$.

Out of these data a matrix \mathcal{D} is defined by

$$\mathcal{D} = (\mathcal{A} \otimes \mathbb{1} - \mathbb{1} \otimes \xi) \oplus \Psi \otimes \mathbb{1} \quad (6)$$

where

$$\mathcal{A} = \begin{pmatrix} A & -B^\dagger \\ B & A^\dagger \end{pmatrix} \quad \Psi = \begin{pmatrix} s & t^\dagger \end{pmatrix} \quad \xi = \begin{pmatrix} \alpha & -\beta^\dagger \\ \beta & \alpha^\dagger \end{pmatrix} \quad (7)$$

The $(2k+n)|\Gamma| \times 2k|\Gamma|$ matrix \mathcal{D} represents the linear map

$$\mathcal{D} : (S^+ \otimes V \otimes \mathcal{T}) \mapsto (Q \otimes V \otimes \mathcal{T}) \oplus (W \otimes \mathcal{T}) \quad (8)$$

To proceed in the KN construction of instantons on X_ζ , one should remember that X_ζ is a smooth resolution of \mathbb{R}^4/Γ . One is thus lead to consider only the Γ -invariant part of (8), \mathcal{D}_Γ . When \mathcal{D} is restricted in this way, it becomes a $(2k+n) \times 2k$ matrix which is the analogue for ALE manifolds of the linear map, $D = a - bx$, appearing in the ADHM construction on \mathbb{R}^4 [19, 28].

Self-duality of the resulting YM connection is imposed by requiring the validity of the ADHM equations

$$\begin{aligned} [A, B] + st &= -\zeta_{\mathbb{C}} \\ \frac{i}{2}([A, A^\dagger] + [B, B^\dagger]) + s^\dagger s - tt^\dagger &= -\zeta_{\mathbb{R}} \\ [\alpha, \beta] &= \zeta_{\mathbb{C}} \\ \frac{i}{2}([\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]) &= \zeta_{\mathbb{R}} \end{aligned} \quad (9)$$

which ensure that $\mathcal{D}_\Gamma^\dagger \mathcal{D}_\Gamma = \Delta \otimes \mathbb{1}$ with Δ a real $k \times k$ matrix and $\mathbb{1}$ the identity

acting in S^+ . We remark that the last two equations in (9) are precisely the equations defining the submanifold $P_\zeta \subset M$ in eq.(4). The instanton bundle on X_ζ is then identified with $\mathcal{E} = \text{Ker}\mathcal{D}_\Gamma^\dagger \subset (\mathcal{Q} \otimes \mathcal{V} \otimes \mathcal{T})_\Gamma \oplus (\mathcal{W} \otimes \mathcal{T})_\Gamma$, where $\mathcal{Q}, \mathcal{V}, \mathcal{W}$ denote the trivial vector bundles over X_ζ with fiber Q, V, W respectively and the subscript Γ means restriction to Γ -invariant subspaces. \mathcal{E} is a complex vector bundle of rank n .

The YM connection on \mathcal{E} is finally given by

$$A_\mu = U^\dagger \nabla_\mu U \quad (10)$$

where U is a $(2k+n) \times n$ matrix of orthonormal sections of $\text{Ker}\mathcal{D}_\Gamma^\dagger$, *i.e.* a matrix obeying $\mathcal{D}_\Gamma^\dagger U = 0$ and $U^\dagger U = \mathbb{1}_{n \times n}$. Since $\mathcal{Q}, \mathcal{V}, \mathcal{W}$ are flat bundles, the covariant derivative on $(\mathcal{Q} \otimes \mathcal{V} \otimes \mathcal{T})_\Gamma$ is simply given by $\nabla_\mu = (\partial_\mu + A_\mu^T)$, with A_μ^T the (self-dual) connection on the tautological bundle with values in \mathcal{H} .

The first and second Chern classes of \mathcal{E} are given by the formulae

$$c_1(\mathcal{E}) = \sum_{i \neq 0} u_i c_1(\mathcal{T}_i) \quad c_2(\mathcal{E}) = \sum_{i \neq 0} u_i c_2(\mathcal{T}_i) + \frac{\dim V}{|\Gamma|} \quad (11)$$

where the vector bundles \mathcal{T}_i are defined by the decomposition $\mathcal{T} = \oplus_i \mathcal{T}_i \otimes R_i$ similar to the decompositions in eq.(5). In terms of the dimensions, w_i and v_i , of the vector spaces W_i and V_i , the integers u_i are defined by $u_i = w_i - C_{ij}v_j$, with $i, j = 0, 1, \dots, r-1$ and C_{ij} is the extended Cartan matrix of the Lie algebra (of type A-D-E) associated to the discrete group Γ .

The complex dimension of the framed moduli space, \mathcal{M}_ζ , which turns out to be a hyperkähler manifold [20,23], is

$$\dim \mathcal{M}_\zeta = \sum_i u_i (v_i + w_i) \quad (12)$$

Generalizing the inverse construction of Corrigan and Goddard [29], Kronheimer

and Nakajima have also shown the uniqueness and completeness of the above construction [20].

We now specialize the discussion to the simplest case of the minimal $SU(2)$ instanton on the EH gravitational background in (1). In this case $\Gamma = \mathbb{Z}_2$, the flat hyperkähler manifold M is \mathbb{R}^8 and the hyperkähler quotient is taken with respect to the group $H = U(1)$, since the two irreducible representations of \mathbb{Z}_2 are one-dimensional ($n_0 = n_1 = 1$). The minimal $SU(2)$ instanton bundle, $\mathcal{E}(k=1, n=2)$, corresponds to the choice $w = (0, 2), v = (0, 1), u = (2, 0)$ and, according to (11), has topological numbers $c_1(\mathcal{E}) = 0, c_2(\mathcal{E}) = \frac{1}{2}$. In this case the matrices A and B , defined in (7), are absent, while

$$\alpha = \begin{pmatrix} 0 & x_1 \\ y_1 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & x_2 \\ y_2 & 0 \end{pmatrix} \quad \Psi = \begin{pmatrix} \sigma_1 & -\bar{\tau}_2 \\ \sigma_2 & \bar{\tau}_1 \end{pmatrix} \quad (13)$$

where x_1, x_2, y_1, y_2 are complex coordinates on \mathbb{R}^8 , $\sigma_1, \sigma_2, \tau_1, \tau_2$ are complex parameters and the bar indicates complex conjugation.

The form of \mathcal{D}^\dagger , when restricted to Γ -invariant subspaces, reduces to

$$\mathcal{D}_\Gamma^\dagger = \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{\sigma}_1 & \bar{\sigma}_2 \\ -y_2 & y_1 & -\tau_2 & \tau_1 \end{pmatrix} \quad (14)$$

Putting $\zeta_{\mathbb{R}} = -a^2, \zeta_{\mathbb{C}} = 0^\star$ and defining $X^2 = |x_1|^2 + |x_2|^2, \rho^2 = |\sigma_1|^2 + |\sigma_2|^2$, the ADHM equations are solved by $y_1 = \lambda x_1, y_2 = \lambda x_2, \tau_1 = \mu \sigma_1, \tau_2 = \mu \sigma_2$ with $\lambda^2 = 1 + \frac{a^2}{X^2}, \mu^2 = 1 - \frac{a^2}{\rho^2}$. In these equations λ and μ have been chosen to be real. This can always be done by exploiting the $U(1)$ isometries of the principal bundles P_ζ and \mathcal{P}_ζ over X_ζ and \mathcal{M}_ζ , respectively. An orthonormal basis for $\text{Ker} \mathcal{D}_\Gamma^\dagger$

★ For our purposes, we are allowed to eliminate the two moduli represented by $\zeta_{\mathbb{C}}$ through a global rotation which corresponds to a (non analytical) change of coordinates on the EH manifold.

can be checked to be

$$U = \frac{1}{X\rho\sqrt{X^2 + \rho^2}} \begin{pmatrix} \rho^2 x_1 & -\mu\rho^2 \bar{x}_2 \\ \rho^2 x_2 & \mu\rho^2 \bar{x}_1 \\ -X^2 \sigma_1 & \lambda X^2 \bar{\sigma}_2 \\ -X^2 \sigma_2 & -\lambda X^2 \bar{\sigma}_1 \end{pmatrix} \quad (15)$$

In this setting, $SU(2)$ gauge transformations correspond to $U \rightarrow UN$ with $N \in SU(2)$. The last ingredient needed to compute the self-dual connection (10) is the abelian ($H = U(1)$) connection on the tautological bundle \mathcal{T} . With a proper gauge choice, $A_\mu^\mathcal{T}$ may be identified with the monopole potential given in (3). Switching to the coordinates employed in (1) and inserting (15) in (10), one explicitly gets

$$A = A_\mu dx^\mu = i \begin{pmatrix} f(r)\sigma_z & g(r)\sigma_- \\ g(r)\sigma_+ & -f(r)\sigma_z \end{pmatrix} \quad (16)$$

where $\sigma_\pm = \sigma_x \pm i\sigma_y$ and

$$f(r) = \frac{t^2 r^2 + a^4}{r^2(r^2 + t^2)} \quad g(r) = \frac{\sqrt{t^4 - a^4}}{r^2 + t^2} \quad (17)$$

with $t^2 = 2\rho^2 - a^2$ and $r^2 = 2X^2 + a^2$.

The connection (16) was previously found in [30] following a completely different procedure. In the limit $a \rightarrow 0$ (16) becomes a connection over $\mathbb{R}^4/\mathbb{Z}_2$ and coincides with the BPST instanton (in the singular gauge) with center at $x_0 = 0$ and size t [28].

The requirement of \mathbb{Z}_2 invariance effectively reduces by a factor of two, in agreement with (12), the dimension of the framed moduli space \mathcal{M}_ζ . It can be checked that \mathcal{M}_ζ exactly coincides with the four-dimensional EH manifold [20,23]. By supersymmetry we also conclude that the number of spinor zero modes in the adjoint representation of the $SU(2)$ gauge group is two, in agreement with the index formulae given in [23].

The explicit expressions of the gaugino and gauge field zero modes can be found starting from the three bounded harmonic scalars of isospin $j = 1$ in the minimal instanton background. In the chosen gauge, which is equivalent to the so-called singular gauge, these correspond to harmonics with angular momentum $l = 0$.

Labelling the components of the scalar isovector by the eigenvalues of j_3 , they obey the equations

$$\begin{aligned}\Delta\theta_{j_3=\pm 1} &= \left[\frac{1}{r^3} \frac{\partial}{\partial r} \left(r^3 \left(1 - \left(\frac{a}{r} \right)^4 \right) \right) \frac{\partial}{\partial r} - (f^2 + g^2) \right] \theta_{j_3=\pm 1} = 0 \\ \Delta\theta_{j_3=0} &= \left[\frac{1}{r^3} \frac{\partial}{\partial r} \left(r^3 \left(1 - \left(\frac{a}{r} \right)^4 \right) \right) \frac{\partial}{\partial r} - 2g^2 \right] \theta_{j_3=0} = 0\end{aligned}\tag{18}$$

whose bounded solutions are

$$\theta_{j_3=0} = \frac{t^2 r^2 + a^4}{t^2(r^2 + t^2)} \quad \theta_{j_3=\pm 1} = \frac{\sqrt{r^4 - a^4}}{r^2 + t^2}\tag{19}$$

The spinor zero-modes in the adjoint representation of $SU(2)$ can now be expressed in the form

$$\lambda_{\alpha(0)}^j = \sigma_{\alpha\dot{\alpha}}^\mu (D_\mu \theta)^j \bar{\epsilon}^{\dot{\alpha}}\tag{20}$$

where $\bar{\epsilon}^{\dot{\alpha}}$ is the covariantly constant spinor on the EH background and $\sigma^\mu = (-i\mathbb{1}, \sigma^j)$, with σ^j the Pauli matrices. Out of the six possible choices of θ^j and $\bar{\epsilon}^{\dot{\alpha}}$, only two linearly independent zero-modes can be obtained. They can be written in terms of a constant spinor η_α as

$$\lambda_{\alpha(0)}^j = f^j \sigma_\alpha^{j\beta} \eta_\beta\tag{21}$$

where the index j is not summed over and

$$f^1 = f^2 = \frac{2(t^2 r^2 + a^4)}{r(r^2 + t^2)^2} \quad f^3 = \frac{2\sqrt{(t^4 - a^4)(r^4 - a^4)}}{r(r^2 + t^2)^2}\tag{22}$$

The norm of the gaugino zero-modes defined in (21) and (22) is computed to be $\sqrt{2}\pi t$.

Instanton-dominated Correlation Functions

We now describe the computation of instanton dominated correlation functions in SYM theory. We recall that the functional integral is performed by expanding the SYM action around its relevant minima up to quadratic terms in the field fluctuations. The quadratic integration gives rise to the determinants of the bosonic and fermionic kinetic operators which, due to the global supersymmetry, cancel each other up to zero-modes [31,32].

According to the number of supersymmetries, we find the following results.

N=1

The $N = 1$ vector multiplet contains a gauge field A_μ^i and a gaugino λ_α^i both in the adjoint representation of the $SU(2)$ gauge group.

The correlation function, which satisfies the chiral selection rule associated to the anomaly of the R-symmetry current in the minimal instanton saddle-point, is $\langle \frac{g^2}{16\pi^2} \text{Tr} \lambda \lambda(x) \rangle$, with g^2 the square of the gauge coupling constant inserted to get a renormalization group invariant answer [1].

The two gaugino insertions are saturated by the two fermionic zero modes (21) and the integration over the bosonic zero-modes is traded for an integration over the four moduli of the minimal instanton. The jacobian for this change of integration variables is

$$\prod_{I=0}^3 \frac{||\delta_I A||}{\sqrt{2\pi}} = \frac{64\pi^4 t^3}{(\sqrt{2\pi}g)^4} \quad (23)$$

where the indices 0, 1, 2, 3 are associated to the collective coordinates $t, \theta^1, \theta^2, \theta^3$, respectively. In detail one has $||\delta_0 A||^2 = \frac{8\pi^2}{g^2} \frac{t^4}{t^4 - a^4}$, $||\delta_1 A||^2 = ||\delta_2 A||^2 = \frac{8\pi^2}{g^2} t^2$ and $||\delta_3 A||^2 = \frac{8\pi^2}{g^2} \frac{t^4 - a^4}{t^2}$. After performing the integration over the fermionic partners, η , of the moduli, which amounts to sum over the permutations of the fermionic

zero-modes, the gaugino condensate is computed to be

$$\begin{aligned} \langle \frac{g^2}{16\pi^2} \text{Tr} \lambda \lambda(x) \rangle &= e^{-\frac{8\pi^2}{g^2} \frac{1}{2}} \mu^3 \int_a^\infty dt \int_{SU(2)/\mathbb{Z}_2} d^3\theta \frac{64\pi^4 t^3}{(\sqrt{2\pi}g)^4} \times \\ & \left(\frac{\sqrt{2}}{\pi t} \right)^2 \frac{g^2}{32\pi^2} \left[\frac{2(t^2 r^2 + a^4)^2}{r^2(r^2 + t^2)^4} + \frac{(t^4 - a^4)(r^4 - a^4)}{r^2(r^2 + t^2)^4} \right] = \frac{1}{2} \Lambda_{N=1}^3 \end{aligned} \quad (24)$$

where $\Lambda_{N=1}$ is the (2-loop) renormalization group invariant scale of the $N = 1$ SYM theory.

N=2

The field content of the $N = 2$ vector multiplet is given by a gauge field A_μ^i , a doublet of gauginos λ_α^{Ai} and a complex scalar ϕ^i , all in the adjoint representation of the gauge group. Apart from the non-anomalous $SU(2)$ global symmetry which rotate the two gauginos, this theory admits an anomalous $U(1)$ R-symmetry under which the λ^A 's have charge +1 and ϕ has charge +2 [33].

In this case we have four gaugino zero modes and the correlation function which is dominated by the minimal instanton is $\langle \text{Tr} \phi^2(x) \rangle$. In fact the lowest order non-trivial contribution is found by expanding the action to second order in the gauge coupling. In this way two powers of the Yukawa interaction term (see eq.(25) below) are brought down from the exponential: each scalar field insertion thus counts as two gaugino zero modes [1]. The next step is to perform the Wick contractions of the scalar fields which effectively amounts to substitute each ϕ^i with $\phi_{(0)}^i$, the solution of the differential equation

$$\Delta \phi_{(0)}^i = \frac{g}{\sqrt{2}} \varepsilon_{jk}^i \varepsilon_{AB} \lambda_{\alpha(0)}^{Aj} \lambda_{(0)}^{\alpha Bk} \quad (25)$$

where ε is the antisymmetric Levi-Civita tensor. In terms of the functions

$$h^1 = h^2 = \frac{\sqrt{(t^4 - a^4)(r^4 - a^4)}}{4(r^2 + t^2)^2} \quad h^3 = -\frac{(a^4 + t^4)r^2 + 2a^4 t^2}{4t^2(r^2 + t^2)^2} \quad (26)$$

the components of $\phi_{(0)}^i$ are

$$\phi_{(0)}^i = -ig\sqrt{2}h^i\eta_A^\alpha(\sigma^i)_\alpha{}^\beta\eta_\beta^A \quad (27)$$

where the index i is not summed over and the η_A^α 's are constant spinors. As before, after performing the integration over the fermionic variables, η_A^α , which implements the sum over the permutations of the gaugino zero-modes, the correlation function becomes

$$\begin{aligned} \langle \text{Tr}\phi^2(x) \rangle &= e^{-\frac{8\pi^2}{g^2}\frac{1}{2}}\mu^2 \int_a^\infty dt \int_{SU(2)/\mathbb{Z}_2} d^3\theta \frac{64\pi^4 t^3}{(\sqrt{2}\pi g)^4} \times \\ & \left(\frac{\sqrt{2}}{\pi t}\right)^4 \left(\frac{\sqrt{2}ig}{4}\right)^2 \left[\frac{((a^4 + t^4)r^2 + 2a^4 t^2)^2}{t^4(r^2 + t^2)^4} + \frac{2(t^4 - a^4)(r^4 - a^4)}{(r^2 + t^2)^4} \right] = -2\Lambda_{N=2}^2 \end{aligned} \quad (28)$$

where $\Lambda_{N=2}$ is the renormalization group invariant scale of the $N = 2$ SYM theory.

N=4

The $N = 4$ supermultiplet contains the gauge vector boson A_μ^i , four gauginos λ^A (in the fundamental representation, **4**, of the $SU(4)$ group which rotates the four SUSY charges) and six scalars ϕ_{AB}^i (in the antisymmetric representation, **6**, of the $SU(4)$ group). All the fields are in the adjoint representation of the $SU(2)$ gauge group and the six scalars satisfy the reality condition: $(\phi_{AB})^* = \phi^{AB} = \frac{1}{2}\varepsilon^{ABCD}\phi_{CD}$. The would-be R-symmetry, which in $N = 1$ notation corresponds to the abelian factor in the manifest $SU(3) \otimes U(1)$ global symmetry, becomes part of the global $SU(4)$ and is not anomalous [33]. In fact, under this conserved $U(1)$ current the gauginos in the three $N = 1$ chiral multiplets have charge +1, while the gaugino in the vector multiplet has charge -3.

For this reason, the simplest correlation function which can receive contribution from the minimal instanton saddle-point is the vacuum expectation value of the identity operator. For SYM theory in flat space, non-perturbative corrections to the vacuum amplitude are expected to be zero [34] and our results point in this

direction (see below). Instead, in curved backgrounds, the vacuum amplitude is conjectured to be the sum of the Euler characteristics of the relevant moduli spaces weighted with the exponential of the corresponding instanton action [5].

The lowest non-trivial order in g at which it is possible to saturate the fermionic zero-modes is g^4 and the functional integral effectively becomes

$$\langle \mathbb{1} \rangle = \frac{g^4}{(\sqrt{2})^4 4!} \langle \left(\int d^4x \varepsilon^{ijkl} \phi_{AB}^i \lambda^{jA} \lambda^{kB} \right)^4 \rangle \quad (29)$$

Performing the Wick contractions between pairs of scalar fields one is left with the scalar propagator acting on the gaugino bilinears. Integrating over two of the four positions amounts to substituting ϕ with the solution (27) of equation (25). After some algebraic manipulations, each one of the remaining two integrands takes the form

$$\sum_i h_i f_{i+1} f_{i+2} \varepsilon_{ABCD} \eta^A \sigma^i \eta^B \eta^C \sigma^i \eta^D = (h_3 f_1 f_2 - h_1 f_2 f_3) \varepsilon_{ABCD} \eta^A \sigma^3 \eta^B \eta^C \sigma^3 \eta^D \quad (30)$$

We immediately notice that for \mathbb{R}^4 this quantity is identically zero, because in this case $f_1 = f_2 = f_3$ and $h_1 = h_2 = h_3$, and we conclude that the vacuum amplitude on \mathbb{R}^4 receives no-correction from the minimal instanton sector. In the EH background one gets instead a non-vanishing result. Integrating (30) gives a result proportional to

$$\int d^4x \sqrt{\det(g^{EH})} (h_3 f_1 f_2 - h_1 f_2 f_3) = -\frac{\pi^2 a^4}{8t^2} \quad (31)$$

where $\det(g^{EH})$ stands for the determinant of the EH metric (1). The integration over the fermionic collective coordinates, η_A , simply gives a factor $4!$. Performing the integral over the moduli one finally gets

$$\langle \mathbb{1} \rangle = e^{-\frac{8\pi^2}{g^2} \frac{1}{2}} \int_a^\infty dt \int_{SU(2)/\mathbb{Z}_2} d^3\theta \frac{64\pi^4 t^3}{(\sqrt{2}\pi g)^4} \left(\frac{\sqrt{2}}{\pi t}\right)^8 \left(\frac{\pi^2 a^4}{8t^2}\right)^2 \left(\frac{4! 3g^4}{4!}\right) = \frac{3}{2} e^{-\frac{8\pi^2}{g^2} \frac{1}{2}} \quad (32)$$

Recalling that \mathcal{M}_ζ coincides with the EH manifold, we see that this result is

consistent with the value of the Euler characteristic of \mathcal{M}_ζ , if one identifies the above result with the “bulk” contribution to $\chi(\mathcal{M}_\zeta)$ which is exactly $3/2$.

Conclusions

In this letter we have reported some preliminary computations of instanton dominated correlation functions in SYM theories on the EH space. We have only studied the minimal instanton case (second Chern class $c_2(\mathcal{E}) = \frac{1}{2}$) leaving the cases of higher Chern class to a forthcoming paper in which the physical and the mathematical aspects of our computations will be discussed at length [22]. From the computations presented here we can already draw some lessons. The constancy of instanton dominated correlation functions of lowest components of chiral superfields seems to persist even on ALE backgrounds. This fact is to be related to the existence of a global supersymmetry charge associated to a covariantly constant Killing spinor.

The independence of gaugino condensates from any scale (*e.g.* a) other than the SYM renormalization group invariant scale bears immediate consequences on the supergravity extension of such computations. A factorization between the gravitational and gauge sectors can in fact be envisaged, making this extension trivial, through a patching of the above computation with those of [32].

The final goal of these considerations is to extract non-perturbative corrections to the low-energy effective field theory emerging from superstring. However, as we remarked before, the minimal instanton is a solution of the superstring classical equation of motion which requires to be adjusted order by order in α' . From this point of view it appears to be more interesting to consider the case in which no α' corrections to the lowest order classical solution are expected, as it happens with the $k = 3$ ($c_2 = \frac{3}{2}$) instanton, thanks to the standard embedding.

In the $N = 2$ case, the calculation performed above should give the first instanton correction to the analytic prepotential of a SYM theory on the EH space. In principle it should be possible to sum up the contributions of gauge instantons

with higher Chern class on ALE spaces, in analogy to what was done in [2].

We would like to stress that all computations presented here were performed with zero expectation value of the scalar fields (we do not expect the constancy of the correlators to be influenced by this choice, unlike, perhaps, the constants in front of the r.h.s of eqs. (24), (28), (32)). The case in which the scalar *vev*'s are different from zero deserve further study.

A last remark concerns the $N = 4$ theory, for which we would like to draw a comparison with the results of [3]. Since we have imposed the same boundary conditions on bosonic and fermionic fields in order to preserve SUSY, our results should be considered as corrections to the Witten index rather than to the free energy.

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